Main results of the diploma

Alternating factor groups of Fuchsian triangle groups

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1 Introduction

This article is the english short version of the german diploma. The german full text can be found on the website `http://www.patrick-reichert.de`. Only the most important chapter is contained in this monograph. It is the last chapter of the original document and it was corrected on several paragraphs.

Fuchsian triangle groups are defined having the presentation

$$\triangle(m_1, m_2, m_3) = \langle x, y \,|\, x^{m_1} = y^{m_2} = (xy)^{m_3} = 1 \rangle,$$

while $m_1, m_2, m_3 > 1$ are integers with $(1/m_1) + (1/m_2) + (1/m_3) < 1$. Every Fuchsian triangle group $\triangle(m_1, m_2, m_3)$ has a faithful representation in $PSL(2, \mathbb{R})$.

For given triangle group $G = \triangle(m_1, m_2, m_3)$ and alternating group A_n this article studies the existence of epimorphisms

$$\varphi\colon \triangle(m_1, m_2, m_3) \mapsto A_n.$$

If φ_1 and φ_2 are two different epimorphisms defined for the same pair (G, A_n), it is interesting whether the kernels

$$N_1 = \ker \varphi_1 \text{ and } N_2 = \ker \varphi_2$$

are conjugate in $PSL(2, \mathbb{R})$.

It will be shown that for every choice of $G = \triangle(m_1, m_2, m_3)$ there is a supergroup H with

$$G \leq H < PSL(2, \mathbb{R})$$

that has the following property: *If N_1, N_2 are normal subgroups of G with finite index and h is an element of $PSL(2, \mathbb{R})$ with $N_1^h = N_2$, then $h \in H$.*

It is due to a famous result of Margulis that the proof of this statement is very simple for non-arithmetic triangle groups. For arithmetic Fuchsian groups an similiar result can be obtained but this article only handles the example

$$\varphi_i\colon \triangle(3, 5, 5) \mapsto A_5.$$

2 Epimorphisms from triangle onto alternating groups

The aim of this section is to look in which cases normal subgroups of Fuchsian triangle groups are conjugate in $PSL(2,\mathbb{R})$. It is due to results of Margulis that for maximal non-arithmetic groups it is easy to answer this question.

2.1 The non-arithmetic case

Looking to epimorphisms that are mapping Fuchsian triangle groups onto alternating groups, it is interesting whether the kernels of these epimorphisms are conjugate in $PSL(2,\mathbb{R})$. In the following section the results of G.A. Margulis will be presented to answer this question for all non-arithmetic triangle groups.

To understand his result some particular properties of triangle groups will be shown. The content of this introduction is based on [GGD99], [SIS03] and [Bea83].

Triangle groups differ from other Fuchsian groups in the following point: All embeddings of a triangle group into the group $PSL(2,\mathbb{R})$ are conjugate. Therefore the notation $\triangle(m_1,m_2,m_3)$ describes a Fuchsian group with signature $(0;m_1,m_2,m_3)$ that is uniquely defined in $PSL(2,\mathbb{R})$ up to conjugacy.

Further the triangle groups have the following remarkable property.

Lemma 2.1 ([Bea83]) *Let G be a discrete group of conformal isometries of the hyperbolic plane. If G contains a triangle group as subgroup, then G itself is a triangle group.*

Definition 2.2 *A **maximal** Fuchsian group is a group, that is not contained in another Fuchsian group.*

The following theorem lists all inclusions between triangle groups and determines thus all maximal triangle groups.

Theorem 2.3 (Singerman, [Sin72]) *This is the complete list of all triangle groups that are contained in another triangle group:*

$$\begin{array}{llll}
\triangle(n,n,n) & \lhd & \triangle(3,3,n) & \textit{with index 3,}\\
\triangle(n,n,n) & \lhd & \triangle(2,3,2n) & \textit{with index 6,}\\
\triangle(n_1,n_1,n_2) & \lhd & \triangle(2,n_1,2n_2) & \textit{with index 2,}\\
\triangle(7,7,7) & < & \triangle(2,3,7) & \textit{with index 24,}\\
\triangle(2,7,7) & < & \triangle(2,3,7) & \textit{with index 9,}\\
\triangle(3,3,7) & < & \triangle(2,3,7) & \textit{with index 8,}\\
\triangle(4,8,8) & < & \triangle(2,3,8) & \textit{with index 12,}\\
\triangle(3,8,8) & < & \triangle(2,3,8) & \textit{with index 10,}\\
\triangle(9,9,9) & < & \triangle(2,3,9) & \textit{with index 12,}\\
\triangle(4,4,5) & < & \triangle(2,4,5) & \textit{with index 6,}\\
\triangle(n,4n,4n) & < & \triangle(2,3,4n) & \textit{with index 6,}\\
\triangle(n,2n,2n) & < & \triangle(2,4,2n) & \textit{with index 4,}\\
\triangle(3,n,3n) & < & \triangle(2,3,3n) & \textit{with index 4,}\\
\triangle(2,n,2n) & < & \triangle(2,3,2n) & \textit{with index 3.}
\end{array}$$

The following definitons and theorems are needed to understand the result of Margulis.

Definition 2.4 *Two Fuchsian groups G_1 and G_2 are said to be **commensurable** if their intersecion $G_1 \cap G_2$ has finite index in both of them.*

Definition 2.5 ([SIS03]) *Let G be a finite covolume Fuchsian group. Define with*

$$Comm(G) = \left\{ t \in PGL(2,\mathbb{R}) \,|\, G \textit{ and } G^t = t^{-1}Gt \textit{ are commensurable} \right\}$$

*the **commensurator** of G. Further define $Comm^+(G)$ to be the subgroup of $Comm(G)$ consisting of conformal elements:*

$$\begin{aligned} Comm^+(G) &= Comm(G) \cap PSL(2,\mathbb{R}) \\ &= \left\{ t \in PSL(2,\mathbb{R}) \,|\, G \textit{ and } G^t \textit{ are commensurable} \right\}. \end{aligned}$$

The importance of the commensurator is because of the following result due to Margulis [Mar91]:

Theorem 2.6 (Margulis)

(1) *Formulation in* [SIS03]*: Let G be a finite covolume Fuchsian group. Then G is a subgroup of finite index in $Comm(G)$ if and only if G is non-arithmetic.*

(2) *Formulation in* [GGD99]*: The commensurator $Comm^+(G)$ of a triangle group G is Fuchsian if and only if G is non-arithmetic.*

Now the announced theorem can be proven as follows.

Theorem 2.7 *Let $G = \triangle(m_1, m_2, m_3) < PSL(2, \mathbb{R})$ be a maximal non-arithmetic triangle group. Then for two different normal subgroups $N_1, N_2 \trianglelefteq G$ with finite index there is no element $h \in PSL(2, \mathbb{R})$ with $N_1^h = N_2$.*

Proof. This is a proof by contradiction. Suppose there is an element $h \in PSL(2, \mathbb{R}) \setminus G$ with $N_1^h = N_2$. It will be shown that this assumption leads to $h \in G$, contrary to the restriction that N_1 and N_2 are different.

Since G is non-arithmetic, theorem 2.6 (1) states, that G is a subgroup of $Comm(G)$ with finite index. While being Fuchsian, G even is contained in $Comm^+(G)$. Using lemma 2.1 it can be obtained that $Comm^+(G)$ is a triangle group. Since G is maximal, $G = Comm^+(G)$ must hold.

N_1 and N_2 are normal subgroups of G with finite index and therefore commensurable. Since they are conjugate in $PSL(2, \mathbb{R})$, the conjugating element h is contained in the commensurator of G. This can be concluded from the following argumentation: Since N_1 and N_2 are normal subgroups of G with finite index, from the inclusions

$$N_2 \leq G \cap G^h \leq G \quad \text{and}$$
$$N_1^h \leq G \cap G^h \leq G^h$$

it can be concluded that $h \in Comm(G)$. Then $h \in PSL(2, \mathbb{R})$ implies the contradiction

$$h \in Comm(G) \cap PSL(2, \mathbb{R}) = Comm^+(G) = G. \quad \square$$

2.2 The arithmetic case

The aim of this section is to look whether the same results like theorem 2.7 can be achieved for arithmetic triangle groups. K. Takeuchi classifies all arithmetic triangle groups of first type in [Tak77a] and [Tak77b]. He shows that they are contained in 18 commensurability classes in the wide sense over 13 fields. The following table contains the full list.

Class	Signature (m_1, m_2, m_3)	Field $\mathbb{F}$
I	(2,4,6) (2,6,6) (3,4,4) (3,6,6)	$\mathbb{Q}$
II	(2,3,8) (2,4,8) (2,6,8) (2,8,8) (3,3,4) (3,8,8) (4,4,4) (4,6,6) (4,8,8)	$\mathbb{Q}(\sqrt{2})$
III	(2,3,12) (2,6,12) (3,3,6) (3,4,12) (3,12,12) (6,6,6)	$\mathbb{Q}(\sqrt{3})$
IV	(2,4,12) (2,12,12) (4,4,6) (6,12,12)	$\mathbb{Q}(\sqrt{3})$
V	(2,4,5) (2,4,10) (2,5,5) (2,10,10) (4,4,5) (5,10,10)	$\mathbb{Q}(\sqrt{5})$
VI	(2,5,6) (3,5,5)	$\mathbb{Q}(\sqrt{5})$
VII	(2,3,10) (2,5,10) (3,3,5) (5,5,5)	$\mathbb{Q}(\sqrt{5})$
VIII	(3,4,6)	$\mathbb{Q}(\sqrt{6})$
IX	(2,3,7) (2,3,14) (2,4,7) (2,7,7) (2,7,14) (3,3,7) (7,7,7)	$\mathbb{Q}(\cos(\pi/7))$
X	(2,3,9) (2,3,18) (2,9,18) (3,3,9) (3,6,18) (9,9,9)	$\mathbb{Q}(\cos(\pi/9))$
XI	(2,4,18) (2,18,18) (4,4,9) (9,18,18)	$\mathbb{Q}(\cos(\pi/9))$
XII	(2,3,16) (2,8,16) (3,3,8) (4,16,16) (8,8,8)	$\mathbb{Q}(\cos(\pi/8))$
XIII	(2,5,20) (5,5,10)	$\mathbb{Q}(\cos(\pi/10))$
XIV	(2,3,24) (2,12,24) (3,3,12) (3,8,24) (6,24,24) (12,12,12)	$\mathbb{Q}(\cos(\pi/12))$
XV	(2,5,30) (5,5,15)	$\mathbb{Q}(\cos(\pi/15))$
XVI	(2,3,30) (2,15,30) (3,3,15) (3,10,30) (15,15,15)	$\mathbb{Q}(\cos(\pi/15))$
XVII	(2,5,8) (4,5,5)	$\mathbb{Q}(\sqrt{2},\sqrt{5})$
XVIII	(2,3,11)	$\mathbb{Q}(\cos(\pi/11))$

The arithmetic group $\triangle(3,5,5)$ has two epimorphisms into the alternating group A_5. The following theorem takes a look to the kernels of these epimorphisms and will be proven in the next sections.

Theorem 2.8 (Main Theorem) *For the triangle group*

$$\triangle(3,5,5) = \langle x, y \mid x^3 = y^5 = (xy)^5 = 1\rangle$$

exist two epimorphisms into the alternating group A_5 defined by

$$\varphi_1(x) = (1,2,3),\ \varphi_1(y) = (1,2,3,4,5),\ \varphi_1(xy) = (1,3,2,4,5) \quad \textit{and}$$
$$\varphi_2(x) = (1,2,4),\ \varphi_2(y) = (1,2,3,4,5),\ \varphi_2(xy) = (1,3,4,2,5).$$

The kernels $N_1 = \ker\varphi_1$ and $N_2 = \ker\varphi_2$ are not conjugate in the group $PSL(2,\mathbb{R})$. Thus the Riemann surfaces $\mathcal{H}^2/N_1$ and $\mathcal{H}^2/N_2$ are two nonisomorphic Riemann surfaces with the same automorphism group A_5 and the same branching type.

The proof of this theorem will consist of the following steps:

- The group $\triangle(3,5,5)$ is contained in the maximal triangle group $\triangle(2,5,6)$. It will be shown that N_1 and N_2 are not conjugate in $\triangle(2,5,6)$.
- Moreover it will be proven that N_1 and N_2 are *surface groups*, i.e. torsion free and with cocompact fundamental regions.
- In the last step it will be concluded that these properties are enough to show that N_1 and N_2 are not conjugate in $PSL(2,\mathbb{R})$.

Embedding of $\triangle(3,5,5)$ into $\triangle(2,5,6)$

Theorem 2.3 states that the triangle group $\triangle(3,5,5)$ is only contained in one different triangle group: as normal subgroup of the maximal arithmetic triangle group $\triangle(2,5,6)$ with index 2. The triangle groups have the following presentations:

$$\triangle(3,5,5) = \langle x_1, y_1 \mid x_1^3 = y_1^5 = (x_1y_1)^5 = 1\rangle,$$
$$\triangle(2,5,6) = \langle x_2, y_2 \mid x_2^2 = y_2^5 = (x_2y_2)^6 = 1\rangle.$$

The following three elements generate a subgroup U of $(2,5,6)$ that is isomorphic to $(3,5,5)$:

(1) The element $x_3 = (x_2y_2)^2$ has order 3.

(2) The element $y_3 = (x_2y_2)^{-2}y_2$ has order 5.

(3) The element $x_3y_3 = y_2$ has order 5.

The order of the element y_3 can be calculated as shown below:

$$\begin{aligned} y_3^5 &= \left[(x_2y_2)^{-2}y_2\right]^5 = \left[y_2^{-1}x_2^{-1}y_2^{-1}x_2^{-1}y_2\right]^5 \\ &= y_2^{-1}x_2^{-1}y_2^{-1}x_2^{-1}y_2y_2^{-1}x_2^{-1}y_2^{-1}x_2^{-1}y_2y_2^{-1}x_2^{-1}y_2^{-1}x_2^{-1}y_2y_2^{-1}x_2^{-1}y_2^{-1}x_2^{-1}y_2y_2^{-1}x_2^{-1}y_2^{-1}x_2^{-1}y_2 \\ &= y_2^{-1}x_2^{-1}(y_2^{-1})^5x_2^{-1}y_2 = y_2^{-1}x_2^{-2}y_2 = 1 \end{aligned}$$

The program GAP has confirmed, that

$$|\triangle(2,5,6) : U| = |\triangle(2,5,6) : \langle (x_2y_2)^2, (x_2y_2)^{-2}y_2\rangle| = 2.$$

It can be shown that there is an element $h \in PSL(2,\mathbb{R})$ with

$$\triangle(3,5,5)^h = U \underset{(2)}{\triangleleft} \triangle(2,5,6).$$

Therefore $\triangle(3,5,5)$ is isomorphic to a subgroup of $\triangle(2,5,6)$. [1]

To decide whether N_1^h and N_2^h are conjugate in $\triangle(2,5,6)$, the normal subgroups N_1 and N_2 have to be embedded into $\triangle(2,5,6)$. This can be done by conjugating each generator of N_i with h or, equivalently, by executing the homomorphism

$$\tau\colon \triangle(3,5,5) \mapsto \triangle(2,5,6),\ \tau(x_1) = x_1^h = (x_2y_2)^2,\ \tau(y_1) = y_1^h = (x_2y_2)^{-2}y_2.$$

This yields the inclusion

$$N_1^h, N_2^h \underset{(60)}{\triangleleft} \triangle(3,5,5)^h \underset{(2)}{\triangleleft} \triangle(2,5,6).$$

Since N_1 and N_2 each have 18 generators, the calculation of N_1^h and N_2^h was done using GAP. The results of the calculations can be summarized as follows:

(1) $N_1^h \triangleleft \triangle(2,5,6),\ \triangle(2,5,6)/N_1^h \cong S_5$.

(2) $N_2^h \triangleleft \triangle(2,5,6),\ \triangle(2,5,6)/N_2^h \ncong S_5$.

(3) $N_1^h \neq N_2^h$ and therefore N_1^h and N_2^h are not conjugate in $\triangle(2,5,6)$.

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[1] In the (german) long version of this diploma a triangle group $\triangle(m_1,m_2,m_3)$ is embedded in $PSL(2,\mathbb{R})$ using the following presentation:

$$\triangle(m_1,m_2,m_3) = \langle x, y \,|\, x^{m_1} = y^{m_2} = (x_1y_1)^{m_3} = 1\rangle,$$

$$x = \begin{pmatrix} -\cos\alpha & -\sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \text{ and } y = \begin{pmatrix} -\cos\beta & -\sin\beta\ q \\ \sin\beta\ \frac{1}{q} & -\cos\beta \end{pmatrix},$$

by choosing the variables as

$$q = \frac{1+B}{1-B},\ B = \sqrt{\frac{\cos(\alpha+\beta)+\cos\gamma}{\cos(\alpha-\beta)+\cos\gamma}},\ \alpha = \frac{\pi}{m_1},\ \beta = \frac{\pi}{m_2},\ \gamma = \frac{\pi}{m_3}\ .$$

The triangle group is contained in the following multiplicative closed set

$$\triangle(m_1,m_2,m_3) = \langle x, y\rangle \subset M_{\triangle(m_1,m_2,m_3)} = \left\{ [a,b,c,d]_{\triangle(m_1,m_2,m_3)} \middle|\ a,b,c,d \in \mathbb{F} \right\},$$

$$[a,b,c,d]_{\triangle(m_1,m_2,m_3)} = \begin{pmatrix} a + b\sin\alpha\sin\beta\ q & c\sin\alpha + d\sin\beta\ q \\ -c\sin\alpha - d\sin\beta\ \frac{1}{q} & a + b\sin\alpha\sin\beta\ \frac{1}{q} \end{pmatrix}$$

over the field $\mathbb{F} = \mathbb{Q}(\cos\alpha, \cos\beta, \cos\gamma)$. Further it is shown, that the conjugating element $h \in PSL(2,\mathbb{R})$ can be expressed as

$$h = \left[\frac{3}{8}\sqrt{3}\left(\sqrt{5}+3\right), -\frac{3}{4}\left(\sqrt{5}+3\right), -\frac{3}{8}\left(\sqrt{5}+1\right), 0\right]_{\triangle(2,5,6)} \in M_{\triangle(2,5,6)}.$$

Torsion free normal subgroups

The aim of this section is to show that N_1 and N_2 are torsion free.

Definition 2.9 *A subgroup N of a Fuchsian group G is called **torsion free** if the unit element is the only element of N of finite order.*

Theorem 2.10 (Corollary 2.10 in [Mag74]) *If N is a normal subgroup of the triangle group*

$$G = \triangle(m_1, m_2, m_3) = \langle x, y \,|\, x^{m_1} = y^{m_2} = (xy)^{m_3} = 1\rangle$$

such that, under the homomorphism $G \mapsto G/N$ the elements x, y, xy of G are respectively mapped onto elements of the same order m_1, m_2, m_3 in G/N, then N is torsion free.

Using this strong result the following corollary can be concluded.

Corollary 2.11 *If p, q, r are primes, then a normal subgroup N of the triangle group $\triangle(p, q, r)$ is torsion free if and only if $x, y, xy \notin N$.*

Proof. Since $x^p = 1$, the equation $(xN)^p = N$ also holds. Because of the assumptions that p is a prime and $xN \neq N$, it follows that xN must have order p. Analogously it can be obtained, that the order of yN must be q and the order of xyN is r. Now theorem 2.10 implies that the normal subgroup N must be torsion free. □

This theorem shows that the normal subgroups N_1 and N_2 are torsion free. The author thanks Prof. G. Rosenberger for the useful email conversation related to this and his submission of an even simpler proof of this statement.

Theorem 2.12 (Private communication with Prof. G. Rosenberger) *The kernel N of every epimorphism $\triangle(3, 5, 5) \mapsto A_5$ is torsion free.*

Proof. This is a proof by contradiction. Suppose that the normal subgroup N is not torsion free. Then N contains an element g of finite order. By theorem 2.10 of [Mag74] g is conjugate to a power of x, y or xy. So three cases must be considered.

If g is conjugate to a power of x, then the element x itself will be contained in N, because N is a normal subgroup. The quotient group G/N is a homomorphic image of $G = \triangle(3, 5, 5)$ and therefore it is generated by the images of the generators x and y:

$$G/N = \langle xN, yN\rangle.$$

For $x \in N$ this reads as $G/N = \langle yN\rangle$. In this case G/N would be cyclic of order 5 and not isomorphic to the alternating group A_5.

In the second case the element g is conjugate to a power of y. Analogously it follows that $y \in N$ and hence $G/N = \langle xN\rangle$. Since G/N would be cyclic of order 3 this case can also not occure.

In the third case g is conjugated to a power of xy. Again $xy \in N$ can be concluded. From $xyN = N$ it follows that $yN = x^{-1}N$ and therefore since $N = (yN)^5 = x^{-5}N$ it follows that $(xN)^5 = 1$. Since the equation $(xN)^3 = 1$ holds, it must be $xN = N$. But this is impossible as shown in the first case above.

By contradiction, it was therefore proved that N is torsion free. This completes the indirect proof of the theorem. □

Surface groups

The aim of this section is to show that N_1 and N_2 are *surface groups.*

Definition 2.13 *A **surface group** is a torsion free Fuchsian group with cocompact fundamental region.*

Because of the last section, it is sufficient to show, that N_1 and N_2 have a cocompact fundamental region. Since they are subgroups of $\triangle(3, 5, 5)$ with finite index, the Riemann-Hurwitz formula states that

$$\mu(N_i) = |\triangle(3, 5, 5) : N_i| \cdot \mu(\triangle(3, 5, 5)) = 60\,\mu(\triangle(3, 5, 5)),$$

whereby $\mu(\Gamma)$ is defined to be the hyperbolic measure of $\mathcal{H}^2/\Gamma$ or equivalently of a fundamental region for Γ. For cocompact groups this measure is finite. Since $\triangle(3, 5, 5)$ is cocompact, every subgroup of finite index must be cocompact, too.

Proof of the main theorem

Definition 2.14 *For $G \leq PSL(2,\mathbb{R})$ define with*

$$N(G) = N_{PSL(2,\mathbb{R})}(G) = \{\alpha \in PSL(2,\mathbb{R}) \,|\, \alpha G \alpha^{-1} = G\}$$

the ***normalizer*** *of G in $PSL(2,\mathbb{R})$.*

Theorem 2.15 (Private communication with Prof. J. Wolfart) *The normalizer of a non-cyclic Fuchsian group is a Fuchsian group.*

Theorem 2.16 (Theorem 9 in [GW04]) *If the $PSL(2,\mathbb{R})$-conjugate surface groups K and K' are both normal subgroups of the triangle group Δ, then $K' = \alpha K \alpha^{-1}$ for some $\alpha \in N(\Delta)$ or $N(\tilde{\Delta})$ where $\tilde{\Delta}$ denotes the normalizer $N(K)$ of K in $PSL(2,\mathbb{R})$.*

For the inclusion $\triangle(3,5,5) \lesssim \triangle(2,5,6)$ this theorem states:

Theorem 2.17 *Let K and K' be two surface groups that are normal subgroups of $\triangle(3,5,5)$ and conjugate in $PSL(2,\mathbb{R})$. Then there exists an element $\alpha \in \triangle(2,5,6) \cup N(N(K))$ with $\alpha K \alpha^{-1} = K'$.*

This theorem can also be expressed as follows:

Theorem 2.18 *Let K and K' be two surface groups that are normal subgroups of $\triangle(3,5,5)$ and that are not conjugate in $\triangle(2,5,6)$. If for every element $\alpha \in N(N(K))$ the condition $\alpha K \alpha^{-1} \neq K'$ holds, the subgroups K and K' are not conjugate in $PSL(2,\mathbb{R})$.*

Lemma 2.19 *If $K \trianglelefteq \triangle(3,5,5) \triangleleft \triangle(2,5,6)$, then $N(N(K)) = \triangle(2,5,6)$.*

Proof. There are two cases to consider.

(a) If $K \triangleleft \triangle(2,5,6)$, then $\triangle(2,5,6) \leq N(K)$ by definition of the normalizer. In this case lemma 2.1 states, that $N(K)$ must be a triangle group. By maximality of $\triangle(2,5,6)$, it follows that $N(K) = \triangle(2,5,6)$. Hence it is $N(N(K)) = N(\triangle(2,5,6)) = \triangle(2,5,6)$ because $\triangle(2,5,6) \leq N(\triangle(2,5,6))$ and $\triangle(2,5,6)$ is maximal.

(b) If $K \ntriangleleft \triangle(2,5,6)$, then at least $\triangle(3,5,5) \leq N(K)$ holds because of $K \triangleleft \triangle(3,5,5)$ and the definition of the normalizer. Using Lemma 2.1 it can be obtained that $N(K)$ is a triangle group. After looking to all inclusions between triangle groups using theorem 2.3 the only possible case is $N(K) = \triangle(3,5,5)$. Further it can be concluded that $N(N(K)) = N(\triangle(3,5,5)) = \triangle(2,5,6)$ as stated because $\triangle(3,5,5) \triangleleft \triangle(2,5,6)$ and in this case the argumentation of part (a) of this proof can be used. □

Using this lemma, the theorem above reads as:

Theorem 2.20 *Let K and K' be two surface groups that are normal subgroups of $\triangle(3,5,5)$. If they are not conjugate in $\triangle(2,5,6)$, then they are not conjugate in $PSL(2,\mathbb{R})$.*

Proof of Main Theorem 2.8. In the last sections it was shown that N_1 and N_2 are surface groups that are not conjugate in $\triangle(2,5,6)$. Using theorem 2.20 it can be concluded that they are also not conjugate in $PSL(2,\mathbb{R})$. □

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